

Invariant Means

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Abstract

Let $m(a, b)$ and $M(a, b, c)$ be symmetric means. We say that M is type 1 invariant with respect to m if $M(m(a, c), m(a, b), m(b, c)) \equiv M(a, b, c)$. If m is strict and isotone, then we show that there exists a unique M which is type 1 invariant with respect to m . In particular we discuss the invariant logarithmic mean L_3 , which is type 1 invariant with respect to $L(a, b) = \frac{b-a}{\log b - \log a}$. We say that M is type 2 invariant with respect to m if $M(a, b, m(a, b)) \equiv m(a, b)$. We also prove existence and uniqueness results for type 2 invariance, given the mean $M(a, b, c)$. The arithmetic, geometric, and harmonic means in two and three variables satisfy both type 1 and type 2 invariance. There are means m and M such that M is type 2 invariant with respect to m , but not type 1 invariant with respect to m (for example, the Lehmer means). L_3 is type 1 invariant with respect to L , but not type 2 invariant with respect to L .

1 Introduction

Let $R_+^n = \{(a_1, \dots, a_n) \in R^n : a_i > 0 \ \forall i\}$. A mean $m(a_1, \dots, a_n)$ in n variables is a continuous function on R_+^n with $\min(a_1, \dots, a_n) \leq m(a_1, \dots, a_n) \leq \max(a_1, \dots, a_n)$. m is called *symmetric* if $m(\pi(a_1, \dots, a_n)) = m(a_1, \dots, a_n)$ for any permutation π . A mean $m(a_1, \dots, a_n)$ is called:

- Strict if $m(a_1, \dots, a_n) = \min(a_1, \dots, a_n)$ or $m(a_1, \dots, a_n) = \max(a_1, \dots, a_n)$ if and only if $a_1 = \dots = a_n$.
- Homogeneous if $m(ka_1, \dots, ka_n) = km(a_1, \dots, a_n)$ for any $k > 0$
- Isotone(strictly) if $m(a_1, \dots, a_n)$ is an increasing(strictly) function of each of its variables.

We let Σ_n denote the set of means in n variables.

Let $A_2(a, b) = \frac{a+b}{2}$ and $A_3(a, b, c) = \frac{a+b+c}{3}$ denote the arithmetic means in two and three variables, respectively. A simple computation shows that

$A_3(A_2(a, c), A_2(a, b), A_2(b, c)) \equiv A_3(a, b, c)$. The same type of invariance holds for the geometric and harmonic means—i.e., $G_3(G_2(a, c), G_2(a, b), G_2(b, c)) \equiv G_3(a, b, c)$ and $H_3(H_2(a, c), H_2(a, b), H_2(b, c)) \equiv H_3(a, b, c)$. The work for this paper started with the aim of finding a mean $M(a, b, c)$ which has the same invariance property with respect to the logarithmic mean in two variables given by

$$L(a, b) = \frac{b - a}{\log b - \log a}$$

We call this mean the invariant logarithmic mean $L_3(a, b, c)$. In general, $M(a, b, c)$ is said to be invariant with respect to $m(a, b)$ if $M(m(a, c), m(a, b), m(b, c)) \equiv M(a, b, c)$. Of course this notion can be extended to means in any number of variables, though we concentrate in this paper on means in two and three variables. We also discuss extensions of some of the other classical means in two variables to invariant means in three or more variables. It is not obvious that such invariant means even exist for a given $m(a, b)$. We prove(Theorem 8) that if m is strict and isotone, then an invariant mean $M(a, b, c)$ does exist. In particular, there is an invariant logarithmic mean in three variables. Various authors have extended the logarithmic mean to three or more variables(see [6], [5], and [4]), but none of those means are invariant. We shall compare the invariant logarithmic mean to those means.

We also show(Proposition 10) that M inherits properties of m such as symmetry, homogeneity, and isotonicity. We say that a symmetric mean $M(a, b, c)$ is invariant if there exists a symmetric $m(a, b)$ such that M is type 1 invariant with respect to m . Not every symmetric mean M is invariant. For example, we show that $M(a, b, c) = \left(\frac{ab + ac + bc}{3} \right)^{1/2}$ is not invariant.

As noted above, the arithmetic, geometric, and harmonic means in three variables are invariant, respectively, with respect to the arithmetic, geometric, and harmonic means in two variables. However, consider the well-known Lehmer means $lh_p(a, b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}}$ and $LH_p(a, b, c) = \frac{a^p + b^p + c^p}{a^{p-1} + b^{p-1} + c^{p-1}}$. It is easy to show that LH_p is *not* invariant with respect to lh_p if $p \neq 1$. However, LH_p is invariant with respect to lh_p in the sense that $LH_p(a, b, lh_p(a, b)) \equiv lh_p(a, b)$. We call this type 2 invariance. The extension of this type of invariance to means in n variables is $m_n(a_1, a_2, \dots, a_{n-1}, m_{n-1}(a_1, \dots, a_{n-1})) \equiv m_{n-1}(a_1, \dots, a_n)$. Let us call the invariance discussed earlier (given by (6)) type 1 invariance. Since the Lehmer means satisfy type 2 invariance, type 2 invariance does not imply type 1 invariance. However, type 1 invariance does not imply type 2 invariance either. For example, $L_3(a, b, c)$ is type 1 invariant with respect to $L(a, b)$, but not type 2 invariant with respect to $L(a, b)$.

Given $M(a, b, c)$, we also prove existence and uniqueness results for type 2 invariance (see Theorem 20 and Theorem 24). We can also prove that given $m(a, b)$, there exists a symmetric mean $M(a, b, c)$ such that M is type 2 invariant with respect to m . Such an M is not unique, however. We *cannot* yet show that if m is analytic, then M can be chosen to be analytic as well.

While the ideas in this paper can also be discussed for non-symmetric means, we usually restrict our results to symmetric means.

2 Preliminary Material

In this section we give some elementary results on means and symmetric functions which will be useful in later sections.

Lemma 1 (1) Let $f(x, y)$ be a differentiable, symmetric function on an open region $E \subset R^2$, and assume that $D = \{(x, y) \in E : x = y\}$ is nonempty. Let i, j be nonnegative integers with $i + j = n$. Then $\frac{\partial^n f}{\partial x^i \partial y^j}(a, a) = \frac{\partial^n f}{\partial x^j \partial y^i}(a, a)$ for any $(a, a) \in D$.

(2) Let $f(x, y, z)$ be a differentiable, symmetric function on an open region $E \subset R^3$, and assume that $D = \{(x, y, z) \in E : x = y = z\}$ is nonempty. Let i, j, k be nonnegative integers with $i + j + k = n$, and let (r, s, t) be any permutation of (i, j, k) . Then $\frac{\partial^n f}{\partial x^i \partial y^j \partial z^k}(a, a, a) = \frac{\partial^n f}{\partial x^r \partial y^s \partial z^t}(a, a, a)$ for any

$(a, a, a) \in D$.

Remark 1 If $Q = (a, a)$, then Lemma 1, part (2) implies, for $n = 2$, that $M_{xy}(Q) = M_{xz}(Q) = M_{yz}(Q)$ and $M_{xx}(Q) = M_{yy}(Q) = M_{zz}(Q)$. If $Q = (a, a, a)$, then for $n = 3$ we have $M_{xxy}(Q) = M_{xxz}(Q) = M_{xyy}(Q) = \dots$ and $M_{xxx}(Q) = M_{yyy}(Q) = M_{zzz}(Q)$.

Proposition 2 Let $m(a, b)$ be a differentiable, symmetric mean, and let $Q = (a, a)$, $a > 0$. Then

- (i) $m_x(Q) = m_y(Q) = \frac{1}{2}$
- (ii) $m_{xy}(Q) = -m_{xx}(Q)$
- (iii) $3m_{xxy}(Q) = -m_{xxx}(Q)$

Proof. The proof follows by taking successively differentiating both sides of $m(x, x) = x$. The details are similar to the proof of Proposition 4 below, and we omit them.

Theorem 3 Let $m(a, b)$ be a differentiable, symmetric, and homogeneous mean, and let $f(x) = m(a, x)$, $a > 0$. Then

- (i) $f'(a) = \frac{1}{2}$
- (ii) $f'''(a) = -\frac{3}{2a}f''(a)$

Proof. (i) follows immediately from Proposition 2, #1. To prove (ii), note first that, since m is homogeneous, $xm_x + ym_y = m$. Taking $\frac{\partial}{\partial x}$ of both sides yields

$$xm_{xx} + ym_{yx} = 0 \tag{1}$$

$\Rightarrow xm_{xx}(x, a) + am_{yx}(x, a) = 0 \Rightarrow m_{yx}(x, a) = -\frac{x}{a}m_{xx}(x, a) = -\frac{x}{a}f''(x)$
 $\Rightarrow m_{yxx}(x, a) = -\frac{1}{a}(xf'''(x) + f''(x)) \Rightarrow m_{yxx}(a, a) = -\frac{1}{a}(af'''(a) + f''(a))$
 $f''(a)$. By Proposition 2 and Lemma 1, $m_{yxx}(a, a) = -\frac{1}{3}f'''(a)$. Setting $-\frac{1}{a}(af'''(a) + f''(a)) = -\frac{1}{3}f'''(a)$ proves (ii).

Proposition 4 *Let $M(a, b, c)$ be a differentiable, symmetric mean, and let $Q = (a, a, a)$, $a > 0$. Then*

- (i) $M_x(Q) = M_y(Q) = M_z(Q) = \frac{1}{3}$
- (ii) $M_{xy}(Q) = -\frac{1}{2}M_{xx}(Q)$
- (iii) $M_{xyz}(Q) = -\frac{1}{2}(M_{xxx}(Q) + 6M_{xxy}(Q))$
- (iv) $M_{xxxx}(Q) + 8M_{xxxy}(Q) + 6M_{xxyy}(Q) + 12M_{xxyz}(Q) = 0$

To prove that $M_x(Q) = \frac{1}{3}$, take $\frac{d}{dx}$ of both sides of the identity $M(x, x, x) = x$. This gives

$$(M_x + M_y + M_z)(x, x, x) = 1 \quad (2)$$

(i) now follows from Lemma 1. Taking $\frac{d}{dx}$ of both sides of (2) and using Clairut's Theorem gives

$$(M_{xx} + M_{yy} + M_{zz} + 2M_{xy} + 2M_{xz} + 2M_{yz})(x, x, x) = 0 \quad (3)$$

Lemma 1 yields $3M_{xx}(Q) + 6M_{xy}(Q) = 0$, which proves (ii). Taking $\frac{d}{dx}$ of both sides of (3) and using Clairut's Theorem again gives
 $(M_{xxx} + M_{yyy} + M_{zzz} + 3(M_{xxy} + M_{xyy} + M_{xxz} + M_{xzz} + M_{yyz} + M_{yzz}) + 6M_{xyz})(Q) = 0$

Lemma 1 then yields (iii). Finally, (iv) follows in a similar fashion.

Proposition 5 *Let $M(a, b, c)$ be a differentiable, symmetric, and homogeneous mean, and let $Q = (a, a, a)$, $a > 0$. Then*

- (i) $M_{xx}(Q) = -a(M_{xxx}(Q) + 2M_{xxy}(Q))$
- (ii) $M_{xxx}(Q) = -\frac{a}{2}(M_{xxxx}(Q) + 2M_{xxxy}(Q))$
- (iii) $M_{xxy}(Q) = -\frac{a}{2}(M_{xxyy}(Q) + M_{xxxy}(Q) + M_{xxyz}(Q))$
- (iv) $M_{xyz}(Q) = -\frac{3a}{2}M_{xxyz}(Q)$

Proof. Since M is homogeneous, $xM_x + yM_y + zM_z = M$. Taking $\frac{\partial}{\partial x}$ of both sides yields

$$xM_{xx} + yM_{yx} + zM_{zx} = 0 \quad (4)$$

Taking $\frac{\partial}{\partial x}$ of both sides of (4) gives

$$xM_{xxx} + M_{xx} + yM_{yxx} + zM_{zxx} = 0 \quad (5)$$

Lemma 1 then implies $M_{xx}(Q) = -a(M_{xxx} + 2M_{xxy})(Q)$, which proves (i).

Taking $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ of both sides of (5) and using Lemma 1 gives (ii) and

(iii). Finally, taking $\frac{\partial}{\partial y}$ of both sides of (4), and then taking $\frac{\partial}{\partial z}$ of both sides yields $xM_{xxyz} + yM_{xyyz} + zM_{xyz} + 2M_{xyz} = 0$. Lemma 1 then implies $3aM_{xxyz} + 2M_{xyz} = 0$, which is (iv).

3 Type 1 Invariance

Definition 6 A symmetric mean $m_n(a_1, \dots, a_n)$ is said to be invariant with respect to $m_{n-1}(a_1, \dots, a_{n-1})$ if

$$m_n(m_{n-1}(a_2, \dots, a_n), m_{n-1}(a_1, a_3, \dots, a_n), \dots, m_{n-1}(a_1, \dots, a_{n-1})) = m_n(a_1, \dots, a_n) \quad (6)$$

for all $(a_1, \dots, a_n) \in R_+^n$.¹

For example, if $A_n(a_1, \dots, a_n) = \frac{a_1 + \dots + a_n}{n}$ and $G_n(a_1, \dots, a_n) = (a_1 \dots a_n)^{1/n}$ denote the arithmetic and geometric means, respectively, in n variables, then for each $n \geq 3$,

$$A_n(A_{n-1}(a_2, \dots, a_n), A_{n-1}(a_1, a_3, \dots, a_n), \dots, A_{n-1}(a_1, \dots, a_{n-1})) \equiv A_n(a_1, \dots, a_n)$$

and

$$G_n(G_{n-1}(a_2, \dots, a_n), G_{n-1}(a_1, a_3, \dots, a_n), \dots, G_{n-1}(a_1, \dots, a_{n-1})) \equiv G_n(a_1, \dots, a_n).$$

For the rest of this section we discuss means $M(a, b, c)$ which are invariant with respect to a given mean $m(a, b)$. In that case, we have

$$M(m(a, c), m(a, b), m(b, c)) = M(a, b, c) \quad (7)$$

Also, given $M(a, b, c)$, if a mean $m(a, b)$ exists with M invariant for m , we say that M is an *invariant mean*. We shall have more to say about this “reverse” process in §5.

¹Later we call this type 1 invariance.

In this section, we are given a mean $m(a, b)$, and we assume throughout that m is a strict, isotone, and symmetric mean. The strictness and isotonicity are necessary in general in order for our proofs to work.

First, given positive numbers a, b, c , define the following recursive sequences:

$$a_{n+1} = m(a_n, c_n), c_{n+1} = m(a_n, b_n), b_{n+1} = m(c_n, b_n), a_0 = a, b_0 = b, c_0 = c \quad (8)$$

We now prove that the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ each converge to the same limit, which lies in the smallest interval containing a, b , and c .

Proposition 7 *Each of the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ converges, and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = L$, with $\min(a_0, b_0, c_0) \leq L \leq \max(a_0, b_0, c_0)$.*

Proof. Assume, without loss of generality, that $a_0 \leq c_0 \leq b_0$. We claim

$$a_k \leq c_k \leq b_k \quad \forall k \quad (9)$$

We prove (9) by induction. So assume that $a_n \leq c_n \leq b_n$ for some $n \geq 0$. Then $a_{n+1} = m(a_n, c_n) \leq m(a_n, b_n) = c_{n+1}$ and $b_{n+1} = m(c_n, b_n) \geq m(a_n, b_n) = c_{n+1}$. That proves (9).

Now for each n , $a_{n+1} \geq a_n$ since $a_n \leq c_n$, and $b_{n+1} \leq b_n$ since $c_n \leq b_n$. Using (9), this implies that $a_n \leq b_n \leq b_0$ and $b_n \geq a_n \geq a_0$. Since $\{a_n\}$ is increasing and bounded above, and $\{b_n\}$ is decreasing and bounded below, $\{a_n\}$ converges to $L_1 \geq a_0$ and $\{b_n\}$ converges to $L_2 \leq b_0$. Since $c_{n+1} = m(a_n, b_n)$, $\{c_n\}$ converges to $m(L_1, L_2) \equiv L_3$. Now $a_{n+1} = m(a_n, c_n) \Rightarrow L_1 = m(L_1, L_3) \Rightarrow L_1 = L_3$ since m is strict. Similarly, $L_2 = L_3$. Letting L equal the common value of L_1, L_2 , and L_3 , each of the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ converges to L , with $a_0 \leq L \leq b_0$. ■

Proposition 7 shows that the recursion (8) defines a mean in three variables $M(a, b, c) = L$, where $a_0 = a, b_0 = b, c_0 = c$, except for the *continuity* of M . We shall prove that in Theorem 13 below. Henceforth we use $M(a, b, c)$ to denote the common limit of the sequences defined by (8). Of course M depends on the given mean m . We prove next that an invariant mean for m is defined precisely in this way.

Theorem 8 *Let L denote the common limit of the sequences defined by (8).*

(A) Define the mean, M , in three variables by $M(a, b, c) = L$, where $a_0 = a, b_0 = b, c_0 = c$. Then M is an invariant mean for m .

(B) If M is an invariant mean for m , then $M(a, b, c) = L$, where $a_0 = a, b_0 = b, c_0 = c$.

Proof. (A) $M(m(a, c), m(a, b), m(c, b))$ = common limit of the sequences defined by (8), with $a_0 = m(a, c), b_0 = m(a, b), c_0 = m(c, b)$. But this is the same limit as that of the sequences defined by (8), with $a_0 = a, b_0 = b, c_0 = c$. Hence $M(m(a, c), m(a, b), m(c, b)) = M(a, b, c)$.

(B) Since M is an invariant mean for m , $M(a_0, b_0, c_0) = M(a_1, b_1, c_1) = \dots = M(a_n, b_n, c_n)$. Taking the limit as $n \rightarrow \infty$ gives $M(a_0, b_0, c_0) = M(L, L, L) = L$. ■

Since the limit of a sequence is unique, Theorem 8 yields the following.

Corollary 9 *Let M_1 and M_2 be invariant means for m . Then $M_1 = M_2$.*

In light of the corollary, we can now speak of *the* invariant mean for m .

Remark 2 *Our approach above is similar in many ways to the well known idea of compounding three given means M_1, M_2, M_3 in three variables to obtain another mean $[M_1, M_2, M_3]$ in three variables (see [7]). Indeed, the invariant mean M can be obtained by compounding the means $M_1(a, b, c) = m(a, c)$, $M_2(a, b, c) = m(a, b)$, and $M_3(a, b, c) = m(b, c)$. However, the standard theorems on compound means do not appear to imply Proposition 7 or Theorem 8. For means in two variables, the existence of the compound mean $[M_1, M_2]$ is proved, for example, in [7] with the assumption that M_1 and M_2 are comparable. A similar assumption is left out of the theorem in [7] for the existence of $[M_1, M_2, M_3]$. However, the proof given also seems to require the assumption of comparability. Note, however, that the means $M_1(a, b, c) = m(a, c)$, $M_2(a, b, c) = m(a, b)$, and $M_3(a, b, c) = m(b, c)$ are **not comparable** in general. .*

Next we show that M inherits many properties of m .

Proposition 10 *Suppose that $M(a, b, c)$ is invariant for $m(a, b)$.*

(A) *Then M is strict, isotone, and symmetric.*

(B) *If m is homogeneous, then M is homogeneous.*

(C) *Suppose that $m_1(a, b) \leq m(a, b) \leq m_2(a, b)$ for all $(a, b) \in R_+^2$, and let M_1 and M_2 be the invariant means for m_1 and m_2 respectively. Then $M_1(a, b, c) \leq M(a, b, c) \leq M_2(a, b, c)$ for all $(a, b, c) \in R_+^3$.*

Proof. The isotonicity of M follows immediately from the isotonicity of m , as does the homogeneity of M if m is homogeneous. To prove the symmetry of M we use (8). If we have the permutation $a_0 \leftrightarrow b_0$, then by (8) and the symmetry of m , $a_n \leftrightarrow b_n$ for all n , and hence the common limit L of the three sequences remains the same. Thus $M(a, b, c) = M(b, a, c)$. If we have the permutation $a_0 \leftrightarrow c_0$, then again by (8) and the symmetry of m , $c_1 \leftrightarrow b_1$, $c_2 \leftrightarrow a_2$, $c_3 \leftrightarrow b_3, \dots$. Hence the sequence $\{a_n\}$ gets sent to the sequence $c_0, a_1, c_2, a_3, c_4, a_5, \dots$, which still converges to L . Thus $M(c, b, a) = M(a, b, c)$. Similarly, one can show that $M(a, c, b) = M(a, b, c)$. Hence $M(\pi(a, b, c)) = M(a, b, c)$ for any permutation π .

Now we prove that M is strict. So suppose that $a_0 < c_0 \leq b_0$. Then $a_1 = m(a_0, c_0) > a_0$ since $m(a, b)$ is strict. This implies that $L > a_0$ since $\{a_n\}$ is increasing by the proof of Proposition 7. Now $c_1 = m(a_0, b_0) \Rightarrow a_0 < c_1 < b_0$ since $m(a, b)$ is strict. Then $b_1 = m(c_0, b_0) \leq b_0$ and $b_2 = m(c_1, b_1) \leq m(c_1, b_0) < b_0$, which implies that $L < b_0$ since $\{b_n\}$ is decreasing, again by the proof of Proposition 7. Thus we have proven (A) and (B).

To prove (C), let $\{a_n^{(k)}\}, \{b_n^{(k)}\}, \{c_n^{(k)}\}$ denote the sequences defined by (8), with $m = m_k$, $k = 1, 2$, and starting with the same initial values $a_0 = a, b_0 = b, c_0 = c$. It is not hard to show, using induction, that $a_n^{(1)} \leq a_n \leq a_n^{(2)}$, $b_n^{(1)} \leq b_n \leq b_n^{(2)}$, and $c_n^{(1)} \leq c_n \leq c_n^{(2)}$. It then follows immediately that $M_1(a, b, c) \leq M(a, b, c) \leq M_2(a, b, c)$.

■

Given $m(a, b)$, we now define the map $\phi : \sum_3 \rightarrow \sum_3$

$$\phi(N)(a, b, c) = N(m(a, c), m(a, b), m(c, b)), \quad N \in \sum_3$$

It follows immediately that a mean $M \in \sum_3$ is a fixed point of ϕ if and only if M is invariant with respect to m . In addition, we now prove that the iterates $\phi^{[n]}(N)$ converge to the invariant mean M .

Theorem 11 *For any $N \in \sum_3$, $\lim_{n \rightarrow \infty} \phi^{[n]}(N)$ exists and equals the invariant mean for m .*

Proof. $\phi^{[n]}(N)(a, b, c) = N(a_n, b_n, c_n)$, where $a_0 = a, b_0 = b, c_0 = c$, and the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ are defined by (8). By Proposition 7, $\lim_{n \rightarrow \infty} \phi^{[n]}(N)(a, b, c) = N(L, L, L) = L$ for each $(a, b, c) \in R_+^3$, where L is the common limit of the three sequences. The Theorem now follows from Theorem 8, Part A. ■

The following result shall prove useful when comparing invariant means to other known means.

Theorem 12 *Let M be the invariant mean for m , and let $N(a, b, c)$ be any mean in three variables. If $N(m(a, c), m(a, b), m(c, b)) \leq (\geq) N(a, b, c)$ for all $(a, b, c) \in R_+^3$, then $M(a, b, c) \leq (\geq) N(a, b, c)$ for all $(a, b, c) \in R_+^3$. Furthermore, the sequence of means $\phi^{[n]}(N)(a, b, c)$ is decreasing(increasing).*

Proof. We prove the \leq case. For any positive integer n , if $\phi(N)(a, b, c) = N(m(a, c), m(a, b), m(c, b)) \leq N(a, b, c)$ for all $(a, b, c) \in R_+^3$, then $\phi^{[2]}(N)(a, b, c) = N(m(m(a, c), m(a, b)), m(m(a, c), m(c, b)), m(m(a, b), m(c, b))) \leq N(m(a, c), m(a, b), m(c, b)) = \phi^{[1]}(N)(a, b, c)$. It follows by successive iteration that $\phi^{[n]}(N)(a, b, c) \leq \phi^{[n-1]}(N)(a, b, c) \leq \cdots \leq N(a, b, c)$ for any positive integer n . Hence $\phi^{[n]}(N)(a, b, c)$ is a decreasing sequence of means. By Theorem 11, taking the limit as n approaches infinity gives $M(a, b, c) \leq N(a, b, c)$.

Now we can prove,

Theorem 13 *M is continuous at each point of R_+^3 .*

Proof. Choose continuous means $N_1(a, b, c)$ and $N_2(a, b, c)$ with $N_1(m(a, c), m(a, b), m(c, b)) \leq N_1(a, b, c)$ and $N_2(a, b, c) \leq N_2(m(a, c), m(a, b), m(c, b))$ for all $(a, b, c) \in R_+^3$ (e.g., $N_1(a, b, c) = \min\{a, b, c\}$ and $N_2(a, b, c) = \max\{a, b, c\}$). Let $f_n = \phi^{[n]}(N_1)$ and $g_n = \phi^{[n]}(N_2)$. Then by Theorem 12, f_n is an increasing sequence of means and g_n is a decreasing sequence of means, each converging pointwise to M . Given $a, b, c > 0$ and $\epsilon > 0$, choose n so that $|g_n(a, b, c) - f_n(a, b, c)| < \frac{\epsilon}{2}$. Choose $\delta > 0$ so that $|f_n(a', b', c') - f_n(a, b, c)| < \frac{\epsilon}{2}$ and $|g_n(a', b', c') - g_n(a, b, c)| < \frac{\epsilon}{2}$ whenever $|a' - a| < \delta$, $|b' - b| < \delta$, and $|c' - c| < \delta$, with $a', b', c' > 0$. Then $M(a', b', c') \leq g_n(a', b', c') \leq g_n(a, b, c) + \frac{\epsilon}{2} < f_n(a, b, c) + \epsilon \leq M(a, b, c) + \epsilon$ and $M(a', b', c') \geq f_n(a', b', c') \geq f_n(a, b, c) - \frac{\epsilon}{2} \geq g_n(a, b, c) - \epsilon \geq M(a, b, c) - \epsilon$, which implies that $|M(a', b', c') - M(a, b, c)| < \epsilon$ whenever $|a' - a| < \delta$, $|b' - b| < \delta$, and $|c' - c| < \delta$, with $a', b', c' > 0$. ■

Theorem 14 *Suppose that $m(a, b)$ is analytic in R_+^2 . Then $M(a, b, c)$ is analytic at $Q = (s, s, s)$ for any $s > 0$.*

Proof. We have that $m(z, w)$ is analytic in a region $D \subset C^2$, with $R_+^2 \subset D$. Let $s > 0$ be given. For $r > 0$, let P_r equal the polydisk $\{(z, w) : |z - s| < r\} \times \{(z, w) : |w - s| < r\}$, and let B_r equal the Euclidean ball $\{(z, w) : \|(z - s, w - s)\|_2 < r\}$. Choose $r > 0$ sufficiently small so that m is analytic at each point of P_r and B_r . Also, for $r > 0$ sufficiently small $|m_z| < \lambda$ and $|m_w| < \lambda$ at all points of B_r , with $\lambda = \frac{1}{\sqrt{2}}$. This is possible since

$m_z(s, s) = m_w(s, s) = \frac{1}{2}$. Let a and b be any points in P_r , and let L be the line segment in C^2 connecting (s, s) to (a, b) . L is given parametrically by $z = at + s(1 - t)$, $w = bt + s(1 - t)$, $0 \leq t \leq 1$. Let $g(t) = m(L(t)) \Rightarrow g(0) = m(L(0)) = m(s, s) = s$. Now $g'(t) = m_z(L(t))(a - s) + m_w(L(t))(b - s)$ $|g'(t)| \leq \lambda \sqrt{(a - s)^2 + (b - s)^2} = \lambda \|(a - s, b - s)\|_2$, $0 \leq t \leq 1$. Since $g(1) - s = \int_0^1 g'(t) dt$,
 $|g(1) - s| \leq \lambda \|(a - s, b - s)\|_2$. Since $g(1) - s = m(a, b) - s$, we have

$$(a, b) \in B_r \Rightarrow |m(a, b) - s| \leq \lambda \|(a - s, b - s)\|_2$$

Now if $(a, b) \in P_r$, then $\|(a - s, b - s)\|_2 \leq \sqrt{2} \|(a - s, b - s)\|_\infty < \sqrt{2}r \Rightarrow |m(a, b) - s| < \lambda \sqrt{2}r = r$. Hence it follows that given any three points a, b, c , with (a, b) , (a, c) , and (b, c) each in P_r , $(m(a, b), m(a, c))$, $(m(a, b), m(b, c))$, and $(m(a, c), m(b, c))$ are also each in P_r . Now for any positive integer n and any mean N analytic in P_r , the sequence of functions $f_n(a, b, c) = \phi^{[n]}(N)(a, b, c)$ is also analytic in P_r . We have also just shown that $\{f_n\}$ is uniformly bounded on P_r . Thus $\{f_n\}$ has a subsequence which converges uniformly on compact subsets of P_r to a function f analytic in P_r (see [1]). Since uniform convergence on compact subsets implies pointwise convergence, Theorem 11 implies that $f(a, b, c) = M(a, b, c)$ at all points of P_r . Therefore M is analytic in P_r , and thus is analytic at Q for any $s > 0$. ■

Remark 3 Using the proof above, one can actually show that M is analytic at any point $(s \pm \epsilon_1, s \pm \epsilon_2, s \pm \epsilon_3)$ for ϵ_1, ϵ_2 , and ϵ_3 sufficiently small and depending on s . One can then enlarge the set of points where M is analytic using the invariance. However, we have not been able to prove that M is analytic at all points of R_+^3 . This is very likely true, but a proof along the lines above has not worked so far.

4 The Invariant Logarithmic Mean

Let $L_3(a, b, c)$ denote the invariant mean for the logarithmic mean $L(a, b) = \frac{b-a}{\log b - \log a}$. Note that L is strict and isotone (see, for example, [3]), so that the results of §3 apply with $m(a, b) = L(a, b)$. The invariance property makes L_3 in some ways a natural generalization of the logarithmic mean in two variables. By Theorem 14 and Proposition 10, L_3 is a strict, isotone, homogeneous mean which is analytic at (a, a, a) for any $a > 0$. Using the iteration (8), one can fairly easily compute $L_3(a, b, c)$ for any $(a, b, c) \in R_+^3$. It is well known (see, for example, [3]) that $G_2(a, b) \leq L(a, b) \leq A_2(a, b)$. Using Proposition 10, it follows immediately that $G_3(a, b, c) \leq L_3(a, b, c) \leq A_3(a, b, c)$. Without this inequality, L_3 would not be a reasonable generalization of L . However, we can obtain tighter upper bounds by considering the means $A_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}$ and $A_p(a, b, c) = \left(\frac{a^p + b^p + c^p}{3}\right)^{1/p}$. It has been shown (see [2]) that $L(a, b) \leq A_{1/3}(a, b)$. Since, for any given p , $A_p(a, b, c)$ is invariant for $A_p(a, b)$, by Proposition 10 again, it follows that $L_3(a, b, c) \leq A_{1/3}(a, b, c) = \left(\frac{a^{1/3} + b^{1/3} + c^{1/3}}{3}\right)^3$, which is a much better bound than $\frac{a+b+c}{3}$. For example, $A_{1/3}(1, 2, 3) \approx 1.87934$, while $L_3(1, 2, 3) \approx 1.87917$.

Stolarsky has defined two generalizations of $L(a, b)$ using second order divided differences (see [4]), $U_0(a, b, c) =$

$$\left(\frac{1}{2} \frac{(a-c)(-c+b)(-b+a)}{a \ln b - a \ln c + (\ln a)c - (\ln b)c - (\ln a)b + (\ln c)b} \right)^{1/2}$$

and $U_1(a, b, c) = \frac{1}{2} \frac{(b-c)(a-c)(a-b)}{a(b-c) \log a - b(a-c) \log b + c(a-b) \log c}.$

U_1 is also a special case of logarithmic means in n variables defined by Pittenger (see [5]), as well as a special case of a family of means defined by the author (see [6]). It is unlikely that U_0 or U_1 are invariant means (in the next section we show that *not all* means in three variables are invariant). However, there is strong evidence for

Conjecture 15 $U_0(a, b, c) \leq L_3(a, b, c) \leq U_1(a, b, c)$ for all $(a, b, c) \in R_+^3$, with equality if and only if $a = b = c$.

To prove Conjecture 15, by Theorem 12, it suffices to prove

Conjecture 16 $U_0(L(a, c), L(a, b), L(c, b)) \geq U_0(a, b, c)$ and $U_1(L(a, c), L(a, b), L(c, b)) \leq U_1(a, b, c)$ for all $(a, b, c) \in R_+^3$.

There is strong numerical evidence that Conjecture 16 is true, but our various attempts at proving it have failed so far.

It is interesting to note that natural generalizations to three variables of certain means $m(a, b)$ are not always comparable to the corresponding invariant mean. For example, consider the Lehmer means $lh(a, b) = \frac{a^2 + b^2}{a + b}$ and $LH(a, b, c) = \frac{a^2 + b^2 + c^2}{a + b + c}$. Note that LH is not invariant for lh . If we let M denote the invariant mean for lh , then $M(1, 2, 3) < LH(1, 2, 3)$, but $M(1, 2, 4) > LH(1, 2, 4)$. Hence M and LH are not comparable.

5 Going in Reverse

In this section we are given a mean $M(a, b, c)$, and we want to discuss the existence and properties of a mean $m(a, b)$ such that M is invariant for m . Recall that if such an m exists, we say that M is an invariant mean. It is natural to ask whether there are any means in three variables which are *not* invariant. We give a simple example shortly of a mean in three variables which is not invariant. First we discuss the analogs of many of the results of §3. We proved in §3 that M inherits many of the properties of m , such as isotonicity and homogeneity. We now prove that m also inherits many of the properties of M , at least when m is analytic. First we need some results about means in general.

We now prove the analog of Corollary 9.

Theorem 17 *Let $M(a, b, c)$ be a differentiable, symmetric mean, and let $m_1(a, b)$ and $m_2(a, b)$ be analytic means, with M invariant for m_1 and for m_2 . Then $m_1 = m_2$.*

Proof. In general, if M is invariant for m , then by letting $c = a$ (or $c \rightarrow a$), we have the relation

$$M(a, m(a, b), m(a, b)) = M(a, a, b) \quad (10)$$

, which holds for all $a, b > 0$. For fixed $a > 0$, let $f(b) = m(a, b)$, $P_1 = (a, m(a, b), m(a, b))$, $P_2 = (a, a, b)$, and $Q = (a, a, a)$.

By successively differentiating (10) k times, one gets an equation of the form $(M_x(P_1) + M_y(P_1))f^{(k)}(b) +$ terms which involve lower order derivatives of f .

For example, differentiating (2) twice² with respect to b gives

$$(M_y + M_z)(P_1)f''(b) + (M_{yy} + 2M_{yz} + M_{zz})(P_1)(f'(b))^2 = M_{zz}(P_2) \quad (11)$$

Letting $b \rightarrow a$ and using Lemma 1, Theorem 3(i), and Proposition 4((i) and (ii)) yields $\frac{2}{3}f''(a) + \frac{1}{4}M_{xx}(Q) = M_{xx}(Q)$, and hence

$$f''(a) = \frac{9}{8}M_{xx}(Q) \quad (12)$$

Differentiating (11) again with respect to b gives

$$(M_y + M_z)(P_1)f'''(b) + 3(M_{yy} + 2M_{yz} + M_{zz})(P_1)f'(b)f''(b) +$$

$$(M_{yyy} + 3M_{yyz} + 3M_{yzz} + M_{zzz})(P_1)(f'(b))^3 = M_{zzz}(P_2) \quad (13)$$

Again, letting $b \rightarrow a$ and using Lemma 1 and Proposition 4((i) and (ii)) yields

$$\frac{2}{3}f'''(a) + \frac{3}{2}M_{xx}(Q)f''(a) - \frac{3}{4}M_{xxx}(Q) + \frac{3}{4}M_{xxy}(Q) = 0 \quad (14)$$

It is clear that we can do this for each k , since we get an equation which can be solved uniquely for $f^{(k)}(a)$, $k = 1, 2, 3, \dots$. If m is analytic, then $f(b) = m(a, b)$ is analytic, and thus this defines $m(a, b)$ uniquely for each fixed a . ■

Theorem 18 *Let $M(a, b, c)$ be a symmetric mean, and let $m(a, b)$ an analytic mean, with M invariant for m . If M is homogeneous, then m is homogeneous.*

Proof. Since M is homogeneous, $M(ka, kb, kc) = kM(a, b, c)$ for any constant $k > 0$. By the invariance property (7), $M(m(ka, kc), m(ka, kb), m(kb, kc)) = kM(m(a, c), m(a, b), m(b, c))$, which implies, again by the homogeneity of M , that $M(\frac{1}{k}m(ka, kc), \frac{1}{k}m(ka, kb), \frac{1}{k}m(kb, kc)) = M(m(a, c), m(a, b), m(b, c))$.

Thus M is invariant for the means $m_1(a, b) = m(a, b)$ and $m_2(a, b) = \frac{1}{k}m(ka, kb)$.

By Theorem 17, $m_1 = m_2$, and thus $m(ka, kb) = km(a, b)$, which means that m is homogeneous.

²One derivative gives no information

5.1 Example

Consider the mean $M(a, b, c) = \left(\frac{ab + ac + bc}{3} \right)^{1/2}$. We shall show that M is not an invariant mean. So let $m(a, b)$ be any mean such that M is invariant for m . Then $M(m(a, b), m(a, b), b) = M(a, b, b)$ for any $b > 0, b \neq a$. This implies that $\left(\frac{m^2(a, b) + 2bm(a, b)}{3} \right)^{1/2} = \left(\frac{b^2 + 2ab}{3} \right)^{1/2} \Rightarrow m^2(a, b) + 2bm(a, b) = b^2 + 2ab \Rightarrow$

$$m(a, b) = -b \pm \sqrt{2(b^2 + ab)} \Rightarrow M(m(1, 2), m(1, 3), m(2, 3)) = \frac{1}{3} \sqrt{(63 - 30\sqrt{3}\sqrt{2} - 36\sqrt{3} + 36\sqrt{2} - 15\sqrt{3}\sqrt{2}\sqrt{5} + 18\sqrt{2}\sqrt{5} + 36\sqrt{5})} \approx 1.9245,$$

 while $M(1, 2, 3) = \frac{1}{3} \sqrt{11} \sqrt{3} \approx 1.9149$. Thus there is no mean $m(a, b)$ such that M is invariant for m . Note that we did not need to assume that m was analytic.

6 Series Expansion

Let $M(a, b, c)$ be invariant with respect to $m(a, b)$, where we assume that M is differentiable, and both symmetric and homogeneous. We would like to expand M in a Taylor Series about $Q = (1, 1, 1)$. Letting $a = 1$ and $f(b) = m(1, b)$, by (12) and Proposition 4(ii)

$$M_{xx}(Q) = \frac{8}{9} f''(1), M_{xy}(Q) = -\frac{4}{9} f''(1) \quad (15)$$

The other second order partials follow from Lemma 1. To compute the third order partials, by Theorem 3(ii), Proposition 4(iii), Proposition 5(i), and (15) we have

$$M_{xxx}(Q) = \frac{32}{27} ((f''(1))^2 - f''(1)), M_{xxy}(Q) = -\frac{4}{27} (4(f''(1))^2 - f''(1))$$

$$M_{xyz}(Q) = \frac{4}{27} (8(f''(1))^2 + f''(1)) \quad (16)$$

Finally we wish to compute the fourth order partials of M . Toward this end, by Proposition 4 and Proposition 5(with $a = 1$), it is easy to show that

$$M_{xxxx}(Q) = -2M_{xxxy}(Q) - 2M_{xxx}(Q) \text{ and } M_{xxyy}(Q) = -M_{xxxy}(Q) - 2M_{xxy}(Q) + \frac{2}{3}M_{xyz}(Q) \quad (17)$$

Differentiating (13) with respect to b gives

$$\begin{aligned}
& (M_y + M_z)(P_1)f''''(b) + 4(M_{yy} + 2M_{yz} + M_{zz})(P_1)f'(b)f'''(b) + \\
& 3(M_{yy} + 2M_{yz} + M_{zz})(P_1)(f''(b))^2 + 6(M_{yyy} + 3M_{yyz} + 3M_{yzz} + M_{zzz})(P_1)(f'(b))^2 f''(b) \\
& + (M_{yyyy} + 4M_{yyyz} + 6M_{yyzz} + 4M_{yzzz} + M_{zzzz})(P_1)(f'(b))^4 = M_{zzzz}(P_2) \quad (18)
\end{aligned}$$

By Lemma 1, Theorem 3, Proposition 4, and Proposition 5, (15), (16), and (17), (18) becomes

$$M_{xxxy}(Q) = -\frac{16}{45}f^{(iv)}(1) - \frac{64}{35}(f''(1))^3 + \frac{448}{405}(f''(1))^2 + \frac{464}{405}f''(1) \quad (19)$$

By (16), and (17) we also have

$$M_{xxxx}(Q) = \frac{32}{45}f^{(iv)}(1) + \frac{128}{35}(f''(1))^3 - \frac{1856}{405}(f''(1))^2 + \frac{32}{405}f''(1)$$

$$M_{xxyy}(Q) = \frac{16}{45}f^{(iv)}(1) + \frac{64}{135}(f''(1))^3 + \frac{352}{405}(f''(1))^2 - \frac{544}{405}f''(1)$$

$$M_{xxyz}(Q) = -\frac{64}{81}(f''(1))^2 - \frac{8}{81}f''(1)$$

Again, the other third and fourth order partials follow from Lemma 1.

Using the above, one can easily compute the Taylor polynomials of orders 2, 3, 4, expanded about $(1, 1, 1)$, for the Invariant Logarithmic Mean

$$L_3(a, b, c). \text{ For example, } T_2(x, y, z) = \frac{1}{3}(x+y+z) - \frac{2}{27}((x-1)^2 + (y-1)^2 + (z-1)^2) + \frac{2}{27}((x-1)(y-1) + (x-1)(z-1) + (y-1)(z-1)).$$

$$\begin{aligned}
T_3(x, y, z) &= T_2(x, y, z) + \frac{28}{729}((x-1)^3 + (y-1)^3 + (z-1)^3) \\
&- \frac{5}{243}((x-1)(y-1)(x+y-2) + (x-1)(z-1)(x+z-2) + (y-1)(z-1)(y+z-2)) \\
&+ \frac{2}{243}(x-1)(y-1)(z-1)
\end{aligned}$$

Note that $T_3(.9, 1, 1.1) = .9977777777777778 \approx .99778$, while $L_3(.9, 1, 1.1) \approx .997771$. However, unless one is fairly close to $(1, 1, 1)$, T_3 and T_4 do not seem to give that good an estimate. For example, $T_3(1, 2, 3) \approx 2.0$ and $T_4(1, 2, 3) \approx 1.71251420732902$, while $L_3(1, 2, 3) \approx 1.8792$; To get a more accurate estimate for $L_3(1, 2, 3)$, it is better to compute $T_4(.5, 1, 1.5)$ and then use the homogeneity property. This gives $L_3(1, 2, 3) = 2L_3(.5, 1, 1.5) \approx 2T_4(.5, 1, 1.5) \approx 1.88014305310602$

7 Another Type of Invariance

In this section we consider the following invariance property for *symmetric* means in two and three variables.

Definition 19 $M(a, b, c)$ is said to be type 2 invariant with respect to $m(a, b)$ if

$$M(a, b, m(a, b)) = m(a, b) \quad (20)$$

for all $(a, b) \in R_+^2$.

Let us call the type of invariance defined earlier as Type 1 invariance. We shall now use the following notation: Write

$(M, m)_j$ if $M(a, b, c)$ is type j invariant with respect to $m(a, b)$, $j = 1, 2$.

It is easy to see that (20) holds for the arithmetic and geometric means in two and three variables. Indeed, let $h(u)$ be any function monotonic on $(0, \infty)$, and define $\overline{m}(a, b) = h^{-1}m(h(a), h(b))$, $\overline{M}(a, b, c) = h^{-1}M(h(a), h(b), h(c))$. It follows that if $(M, m)_2$, then $(\overline{M}, \overline{m})_2$. In particular, $h^{-1}\left(\frac{h(a) + h(b) + h(c)}{2}\right)$ is

type2 invariant with respect to $h^{-1}\left(\frac{h(a) + h(b)}{2}\right)$ for each monotone h .

Earlier we noted that if $lh(a, b) = \frac{a^2 + b^2}{a + b}$ and $LH(a, b, c) = \frac{a^2 + b^2 + c^2}{a + b + c}$, then LH is not type 1 invariant with respect to lh . However, it is easy to show that LH is type 2 invariant with respect to lh . It is natural to ask whether type 1 invariance is stronger than type 2 invariance, i.e., does type 1 invariance imply type 2 invariance? The answer can be seen by looking at the invariant logarithmic mean $L_3(a, b, c)$ defined earlier. If $L(a, b)$ denotes the logarithmic mean, then $L_3(1, 2, L(1, 2)) \approx 1.442708$, while $L(1, 2) \approx 1.442695$.

While $L_3(1, 2, L(1, 2))$ and $L(1, 2)$ are close, it appears that they are not equal. We shall prove this shortly using a series expansion, and thus L_3 is not type 1 invariant with respect to L .

Remark 4 *Unlike type 1 invariance, m symmetric does not necessarily imply that M is symmetric.*

Remark 5 *As with type 1 invariance, one can attempt to view the invariant mean M as a special case of compounding three means in three variables by letting $M_1(a, b, c) = a$, $M_2(a, b, c) = b$, and $M_3(a, b, c) = m(a, b)$. However, the limit in this case does not exist.*

7.1 Given $M(a, b, c)$

We now prove an *existence* result for type 2 invariance, given $M(a, b, c)$.

Theorem 20 *Let $M(a, b, c)$ be an isotone, symmetric mean. Then there exists a symmetric mean $m(a, b)$ such that $(M, m)_2$.*

Proof. Let $n(a, b)$ be any symmetric mean, and let a and b be given positive numbers. Let $g(z) = M(a, b, z)$, and define the recursive sequence

$c_{k+1} = g(c_k)$, $c_0 = n(a, b)$. Since M is isotone, a simple inductive proof shows that $\{c_k\}$ is decreasing if $c_1 \leq c_0$, while $\{c_k\}$ is increasing if $c_1 \geq c_0$. Since $|c_k| \leq \max\{a, b\}$, in either case $\{c_k\}$ is bounded and monotonic, and hence converges to some real number L , $\min\{a, b\} \leq L \leq b \leq \max\{a, b\}$. Note that if $c_1 = c_0$, then $c_k = c_0$ for all k , and thus $\{c_k\}$ converges to $L = c_0$. Define the mean $m(a, b) = L$. Since M and n are symmetric, $m(b, a) = m(a, b) \Rightarrow m$ is symmetric. Of course the iteration $c_{k+1} = g(c_k)$ converges to a fixed point of g , and thus $g(L) = L$. This implies that $M(a, b, m(a, b)) = m(a, b)$, which proves that $(M, m)_2$. The only thing left to prove is that m is continuous. Toward this end, let $\{n_k\}$ be the sequence of means defined by the recursion $n_{k+1}(a, b) = M(a, b, n_k(a, b))$, $n_0(a, b) = n(a, b)$. Note that since M and n are continuous, each n_k is also continuous. It is easy to show that, for each fixed a and b , $c_{k+1} = n_k(a, b)$. Now let $g_k(a, b)$ be the sequence of means corresponding to $n(a, b) = \min\{a, b\}$, and let $h_k(a, b)$ be the sequence of means corresponding to $n(a, b) = \max\{a, b\}$. If $c_0 = \min\{a, b\}$, then $c_1 \leq c_0$ and thus $\{c_k\}$ is decreasing. Hence g_k is a decreasing sequence of means converging to $m(a, b)$. Similarly, h_k is an increasing sequence of

means converging to $m(a, b)$. The rest of the proof now follows in a similar fashion to the proof of Theorem 13. Given $a, b > 0$ and $\epsilon > 0$, choose k so that $|g_k(a, b) - f_k(a, b)| < \frac{\epsilon}{2}$. Choose $\delta > 0$ so that $|g_k(a', b') - g_k(a, b)| < \frac{\epsilon}{2}$ and $|h_k(a', b') - h_k(a, b)| < \frac{\epsilon}{2}$ whenever $|a' - a| < \delta$, and $|b' - b| < \delta$, with $a', b' > 0$. Then $m(a', b') \leq g_k(a', b') \leq g_k(a, b) + \frac{\epsilon}{2} < h_k(a, b) + \epsilon \leq m(a, b) + \epsilon$ and

$m(a', b') \geq h_k(a', b') \geq h_k(a, b) - \frac{\epsilon}{2} \geq g_k(a, b) - \epsilon \geq m(a, b) - \epsilon$, which implies that $|m(a', b') - m(a, b)| < \epsilon$ whenever $|a' - a| < \delta$ and $|b' - b| < \delta$, with $a', b' > 0$. ■

One may of course discuss type 2 invariance for functions $M(a, b, c)$ and $m(a, b)$ which are not necessarily means. Our next result shows, however, that if M is a mean, then m must also be a mean.

Lemma 21 *Suppose that $M(a, b, c)$ is a strict mean, and that $g(a, b)$ is any symmetric continuous function satisfying $M(a, b, g(a, b)) = g(a, b)$ for all $(a, b) \in R_+^2$. Then g is a strict symmetric mean.*

Proof. Suppose that $a < b$ and that $g(a, b) \leq a$. Since M is a strict mean, $M(a, b, g(a, b)) > g(a, b) = M(a, b, g(a, b))$, a contradiction. Hence $g(a, b) > a$ whenever $a < b$. Similarly, $g(a, b) < b$ whenever $a < b$, and thus g is a strict mean. ■

Lemma 22 *Suppose that $M(a, b, c)$ is a strictly isotone mean, and that $m(a, b)$ is a mean with $(M, m)_2$. Then m is a strictly isotone mean.*

Proof. Let $a > 0$ and suppose that $b_1 < b_2$ with $m(a, b_1) = m(a, b_2)$. Since $(M, m)_2$, this implies that $M(a, b_1, m(a, b_1)) = M(a, b_2, m(a, b_1))$, which contradicts the fact that M is strictly isotone. Hence $m(a, b)$ is either increasing or decreasing in b . Now for $a \leq b$, $a = m(a, a)$ and $a \leq m(a, b)$. Thus $m(a, b)$ must be increasing in b for any fixed $a > 0$. Similarly, $m(a, b)$ must be increasing in a for any fixed $b > 0$.

We have not been able to prove a *uniqueness* result for type 2 invariance. That is, does $(M, m_1)_2 = (M, m_2)_2$ imply that $m_1 = m_2$? We can prove uniqueness with the additional assumption that M_{zz} does not change sign.

Remark 6 *The proofs given above show that Lemmas 22, 21, and Theorem 24 are all local results. That is, one need only assume that the hypotheses hold for all $(a, b) \in I_1 \times I_2$, where I_1 and I_2 are open subintervals of the positive reals. The conclusions then also hold for all $(a, b) \in I_1 \times I_2$.*

Proposition 23 *Let $M(a, b, c)$ be a symmetric, strictly isotone mean which is twice differentiable in R_+^3 . Assume also that $M_{zz}(x, y, z)$ is never 0 on R_+^3 , and let $m(a, b)$ be a mean with $(M, m)_2$. Then $f(b) = m(a, b)$ is differentiable for each fixed $a > 0$, and $M_z(a, b, m(a, b)) < 1$ for each $a, b > 0$.*

Proof. $M_y(a, b, m(a, b)) \geq 0$ and $M_z(a, b, m(a, b)) \geq 0$ since M is strictly isotone. Note that $M_{yy}(a, b, c) = M_{zz}(a, c, b)$ since M is symmetric. Hence $M_{yy}(a, b, c)$ is never 0 on R_+^3 . This easily implies the strict positivity of M_y —i.e., $M_y(a, b, c) > 0$ for all $(a, b, c) \in R_+^3$. By Theorem 20, there exists a symmetric mean $m(a, b)$ such that $(M, m)_2$. Let $f(b) = m(a, b)$ for each fixed $a > 0$. Since f is increasing by Lemma 22, $f'(b)$ exists on a set $S \subset (0, \infty)$, with $m(S^c) = 0$, where S^c denotes the complement of S . Differentiating both sides of (20) with respect to b gives

$$M_y(a, b, m(a, b)) + M_z(a, b, m(a, b))f'(b) = f'(b) \quad (21)$$

Note that (21) implies that $f'(b) > 0$ on S . (21) also implies that, for $b \in S$, $M_y(a, b, f(b)) + M_z(a, b, f(b))f'(b) > M_z(a, b, f(b))f'(b) \Rightarrow f'(b) > M_z(a, b, f(b))f'(b)$. Hence we have $M_z(a, b, f(b)) < 1$ whenever $b \in S$. Solving (21) for $f'(b)$ yields $f'(b) = \frac{M_y(a, b, f(b))}{1 - M_z(a, b, f(b))}$. This shows that $f''(b)$ exists (indeed $f^{(k)}(b)$ exists for any k). Hence, if $b \in S$, we can differentiate both sides of (21) with respect to b to obtain

$$M_{yy}(P) + 2M_{yz}(P)f'(b) + M_z(P)f''(b) + M_{zz}(P)(f'(b))^2 = f''(b) \quad (22)$$

where $P = (a, b, f(b))$. Thus $M_{yy}(P) + 2M_{yz}(P)f'(b) + M_{zz}(P)(f'(b))^2 = (1 - M_{zz}(P))f''(b)$. Upon dividing thru by $f'(b)$ we have

$$\frac{M_{yy}(P)}{f'(b)} + 2M_{yz}(P) + M_{zz}(P)f'(b) = (1 - M_z(P))\frac{f''(b)}{f'(b)} \quad (23)$$

Now suppose that $\{b_n\}$ is a sequence in S , with $b_n \rightarrow b^+$, $b \notin S$. If $f'(b_n) \rightarrow r^+$, then by (21), $r = M_y(a, b, f(b)) + M_z(a, b, f(b))r > M_z(a, b, f(b))r \Rightarrow M_z(a, b, f(b)) < 1$. The same conclusion follows if $b_n \rightarrow b^-$. By taking convergent subsequences, we can now conclude that

If $b_n \in S, b \notin S, b_n \rightarrow b^+ \text{ or } b^-$, and $f'(b_n) \nrightarrow \infty$, then $M_z(a, b, f(b)) < 1$

We now prove that $f'(b_n)$ cannot approach ∞ . So suppose that $b_n \in S, b_n \rightarrow b_0$ and $f'(b_n) \rightarrow \infty$. Assume first that $M_{zz}(a, b, c) > 0$ on R_+^3 . Then, upon replacing b by b_n , the LHS of (23) approaches ∞ as $n \rightarrow \infty$. Thus the LHS of (23) is positive for n sufficiently large. Since $M_z(a, b_n, f(b_n)) < 1$, the RHS of (23) implies that $f''(b_n) > 0$ for n sufficiently large. If $M_{zz}(a, b, c) < 0$ on R_+^3 , then a similar argument shows that $f''(b_n) < 0$ for n sufficiently large. Now if $f'(b_n) \rightarrow \infty$ as $b_n \rightarrow b_0$ from both sides, then f must be concave to one side of b_0 and convex on the other side of b_0 . We have just shown that that is impossible, and thus $f'(b_n) \nrightarrow \infty$ as $b_n \rightarrow b_0$. We can now conclude that

$$M_z(a, b, m(a, b)) < 1 \text{ for all } a, b > 0 \quad (24)$$

Finally, let $c = m(a, b)$, $\Delta c = m(a, b + \Delta b) - m(a, b)$. By the Mean Value Theorem, $M(a, b + \Delta b, c + \Delta c) - M(a, b, c) =$

$M_y(a, b + t \Delta b, c + t \Delta c) \Delta b + M_z(a, b + t \Delta b, c + t \Delta c) \Delta c$, $0 < t < 1$. Also, since $(M, m)_2$, $M(a, b + \Delta b, c + \Delta c) - M(a, b, c) = M(a, b + \Delta b, m(a, b + \Delta b)) - M(a, b, m(a, b)) =$

$$m(a, b + \Delta b) - m(a, b) = \Delta c. \text{ Hence } \frac{\Delta c}{\Delta b} = M_y(a, b + t \Delta b, c + t \Delta c) \\ c) + M_z(a, b + t \Delta b, c + t \Delta c) \frac{\Delta c}{\Delta b} \Rightarrow \\ \frac{\Delta c}{\Delta b} = \frac{M_y(a, b + t \Delta b, c + t \Delta c)}{1 - M_z(a, b + t \Delta b, c + t \Delta c)}. \text{ Letting } \Delta b \rightarrow 0, (24) \text{ implies that}$$

$f'(b)$ exists and equals $\frac{M_y(a, b, c)}{1 - M_z(a, b, c)}$ whenever $M_z(a, b, m(a, b)) \neq 1$. ■

Theorem 24 *Let $M(a, b, c)$ be a symmetric, strictly isotone mean which is twice differentiable in R_+^3 . Assume that $M_{zz}(x, y, z)$ is never 0 on R_+^3 . If m_1 and m_2 are means with $(M, m_1)_2 = (M, m_2)_2$, then $m_1 = m_2$.*

Proof. Let $g(z) = M(a, b, z)$ for fixed $a > 0, b > 0$. If $M(a, b, m_1(a, b)) = m_1(a, b)$ and $M(a, b, m_2(a, b)) = m_2(a, b)$, then $g(L_1) = L_1$ and $g(L_2) = L_2$, where $L_j = m_j(a, b)$. If $m_1(a, b) \neq m_2(a, b)$, then L_1 and L_2 are distinct fixed points of g . Now $\frac{g(L_2) - g(L_1)}{L_2 - L_1} = 1 \Rightarrow g'(t) = 1$ for some $t, L_1 < t < L_2$.

By Proposition 23, $g'(L_1) \leq 1$ and $g'(L_2) \leq 1$. Since g cannot be constant, this implies that g' must be increasing somewhere on (L_1, L_2) and decreasing somewhere on (L_1, L_2) . That contradicts the fact that g'' is never 0 on $(0, \infty)$. Hence $m_1(a, b) = m_2(a, b)$ for all $a > 0, b > 0$. ■

We now prove the existence of an analytic m such that $(M, m)_2$.

Theorem 25 *Let $M(a, b, c)$ be a symmetric, strictly isotone mean which is analytic in R_+^3 . Assume also that $M_{zz}(x, y, z)$ is never 0 on R_+^3 . Then there exists a unique symmetric mean $m(a, b)$ which is analytic in R_+^2 , and such that $(M, m)_2$.*

Proof. Let $T(z_1, z_2, z_3) = M(z_1, z_2, z_3) - z_3$, which is analytic in some open set in C^3 containing R_+^3 . Then, for all $x > 0$, $y > 0$, $T(x, y, m(x, y)) = 0$ and $T_{z_3}(x, y, m(x, y)) \neq 0$ by Proposition 23. By the Implicit Function Theorem (see [1]), for any given $a > 0$, $b > 0$, the equation $T(z_1, z_2, z_3) = 0$ along with $T(a, b, m(a, b)) = 0$ has a unique solution $z_3 = g(z_1, z_2)$ analytic in some open neighborhood O of (a, b) in C^2 , with $g(a, b) = m(a, b)$. Restricting z_1 and z_2 to be real, we have $M(x, y, g(x, y)) = g(x, y)$ for all $(x, y) \in I = O \cap R_+^2$. Note that since $M(z_2, z_1, z_3) - z_3 = M(z_1, z_2, z_3) - z_3$ and $m(b, a) = m(a, b)$, by uniqueness $g(z_2, z_1) = g(z_1, z_2)$. By Lemma 21 (see also Remark 6), $g(x, y)$ must be a symmetric mean, at least for $(x, y) \in I$. Since $M(x, y, m(x, y)) = m(x, y)$ for all $(x, y) \in R_+^2$, Theorem 24 (see also Remark 6) then implies that $m(x, y) = g(x, y)$ for all $(x, y) \in I$. Hence m extends to be analytic in an open neighborhood of any $(a, b) \in R_+^2$. The uniqueness of m follows from Theorem 24. ■

Example 26 *Let $M(a, b, c) = \left(\frac{ab + ac + bc}{3} \right)^{1/2}$, which interpolates the arithmetic and geometric means in three variables. Then*

$$M_{zz} = -\frac{1}{4} \frac{(x+y)^2}{(xy+xz+yz)\sqrt{(3xy+3xz+3yz)}}, \text{ which is never 0 on } R_+^3.$$

Hence, by Theorem 25, there exists a unique symmetric mean $m(a, b)$ which

is analytic in R_+^2 , and such that $(M, m)_2$. Let $g(a, b) = \left(\frac{a^{1/2} + b^{1/2}}{2} \right)^2$, which equals the Holder mean $A_{1/2}(a, b)$. Let $h(a, b) = M(a, b, g(a, b)) =$

$$\frac{1}{6} \sqrt{(18ab + 3a^2 + 6(\sqrt{a})^3 \sqrt{b} + 6(\sqrt{b})^3 \sqrt{a} + 3b^2)}.$$

$$\text{Now } h(1, b) = \frac{1}{6} \sqrt{(18b + 3 + 6\sqrt{b} + 6(\sqrt{b})^3 + 3b^2)} \text{ and } g(1, b) = \frac{1}{4} (1 + \sqrt{b})^2.$$

Then $h(1, b) > g(1, b) \Leftrightarrow$

$\frac{1}{36}(3u^4 + 6u^3 + 18u^2 + 6u + 3) > \frac{1}{16}(1 + u)^4$, where $u = \sqrt{b}$. The latter inequality holds if and only if $(u - 1)^4 > 0$, which holds for any $u \neq 1$. Hence $h(1, b) > g(1, b)$ for any $b \neq 1$. Since g and h are each homogeneous of degree 1, $h(a, b) > g(a, b)$ for all $a, b > 0$ with $a \neq b$. We have shown that

$M(a, b, g(a, b)) \geq g(a, b)$, with equality if and only if $a = b$. It follows that if $\{n_k\}$ is the sequence of means defined by the recursion $n_{k+1}(a, b) = M(a, b, n_k(a, b))$, $n_0(a, b) = g(a, b)$, then $\{n_k(a, b)\}$ is increasing and converges to $m(a, b)$ for each $a, b > 0$ (see the proof of Theorem 20). Since $n_1(a, b) = h(a, b) > g(a, b) = n_0(a, b)$ for all $a, b > 0$ with $a \neq b$, we have proven that $m(a, b) > \left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2$, with equality if and only if $a = b$.

We now prove a uniqueness result without most of the assumptions on M in Theorem 24. However, we then must assume that m_1 and m_2 are analytic.

Theorem 27 *Let $M(a, b, c)$ be a differentiable, symmetric mean, and let $m_1(a, b)$ and $m_2(a, b)$ be symmetric means, each analytic in R_+^2 . Then $(M, m_1)_2 = (M, m_2)_2$ implies that $m_1 = m_2$.*

Proof. If $M(a, b, c)$ is type 2 invariant with respect to $m(a, b)$, then

$$M(a, b, m(a, b)) = m(a, b) \quad (25)$$

Letting $f(b) = m(a, b)$ and differentiating both sides of (22) with respect to b gives

$$\begin{aligned} M_z(P)f'''(b) + 3M_{yz}(P)f''(b) + 3M_{zz}(P)f'(b)f''(b) + 3M_{yyz}(P)f'(b) \\ + 3M_{yzz}(P)(f'(b))^2 + M_{yyy}(P) + M_{zzz}(P)(f'(b))^3 = f'''(b) \end{aligned} \quad (26)$$

In general, taking k derivatives of (25) with respect to b yields an equation of the form $M_z(P)f^{(k)}(b) + A = f^{(k)}(b)$, where A is a polynomial in the partial derivatives of M (evaluated at P) and derivatives of f (evaluated at b) of order $< k$. Since $M_z(a, a, a) = \frac{1}{3}$, letting $b \rightarrow a$, we get an equation which can be solved uniquely for $f^{(k)}(a)$, $k = 1, 2, 3, \dots$. If m is analytic, then $f(b) = m(a, b)$ is analytic, and thus this defines $m(a, b)$ uniquely for each fixed a . ■

7.2 Series Expansion

Letting $b \rightarrow a$ in (22), and using $f'(a) = \frac{1}{2}$ and $M_z(Q) = \frac{1}{3}$ yields $M_{yy}(Q) + M_{yz}(Q) + \frac{1}{4}M_{zz} = \frac{2}{3}f''(a)$, where $Q = (a, a, a)$. Since M is symmetric, Lemma 1 implies $\frac{5}{4}M_{xx}(Q) + M_{xy}(Q) = \frac{2}{3}f''(a)$. Finally, applying Proposition 4 yields

$$M_{xx}(Q) = \frac{8}{9}f''(a) \quad (27)$$

, which is just (14).

Letting $b \rightarrow a$ in (26) and using Lemma 1, Proposition 4(ii), and Proposition 5(i) yields $-\frac{9}{8a}M_{xx}(Q) = \frac{2}{3}f'''(a)$. By (27) we have $f'''(a) = -\frac{3}{2a}f''(a)$, which already holds by Theorem 3. So to get new information we must differentiate both sides of (26) again with respect to b . Then letting $b \rightarrow a$ and using Lemma 1 gives

$$\begin{aligned} & 4M_{yz}(Q)f'''(a) + 2M_{zz}(Q)f'''(a) + 3M_{zz}(Q)(f''(a))^2 + 12M_{yyz}(Q)f''(a) \\ & + \frac{3}{2}M_{zzz}(P)f''(a) + \frac{3}{2}M_{yyzz}(Q) + \frac{5}{2}M_{yyyz}(Q) + \frac{17}{16}M_{xxxx}(Q) = \frac{2}{3}f^{(iv)}(a) \end{aligned} \quad (28)$$

Since we are interested in the series expanded about $(1, 1, 1)$, we now let $a = 1$. By (17), Theorem 3, and (27), (28) becomes

$$\frac{8}{3}(f''(1))^3 - \frac{16}{3}(f''(1))^2 + \frac{8}{3}f''(1) + \frac{3}{8}M_{xxx}(Q) - \frac{9}{2}f''(1)M_{xxx}(Q) - \frac{9}{8}M_{xxx}(Q) = \frac{2}{3}f^{(iv)}(1) \quad (29)$$

7.3 Given $m(a, b)$

We now prove a result similar to Theorem 20, except here we are given the mean $m(a, b)$.

Theorem 28 *Let $m(a, b)$ be a symmetric mean. Then there exists a symmetric mean $M(a, b, c)$ such that $(M, m)_2$.*

Proof. Let $n(a, b)$ be any mean in two variables (not necessarily symmetric), and let $M(a, b, c) = n(m(a, b), c)$, for c between a and b . Then extend $M(a, b, c)$ to the rest of R_+^3 so that M is symmetric. It follows that

$$M(a, b, c) = \begin{cases} n(m(a, b), c) & \text{if } a \leq c \leq b \text{ or } b \leq c \leq a \\ n(m(a, c), b) & \text{if } a \leq b \leq c \text{ or } c \leq b \leq a \\ n(m(c, b), a) & \text{if } b \leq a \leq c \text{ or } c \leq a \leq b \end{cases} \quad (30)$$

Since m is symmetric, it is not hard to see that M must also be symmetric. It also follows easily that M is continuous. For example, fix $a \neq c$, and let $b \rightarrow c$. Using the definition of M above (first two rows), $n(m(a, b), c)$ and $n(m(a, c), b)$ each approach $n(m(a, c), c)$ by the continuity of m and of n , respectively. Finally, since m is a mean, if $c = m(a, b)$, then $a \leq c \leq b$ or $b \leq c \leq a$. Hence $M(a, b, m(a, b)) = n(m(a, b), m(a, b)) = m(a, b)$. Thus $(M, m)_2$.

Remark 7 *If $m(a, b)$ is analytic, we have not been able to prove the existence of a symmetric analytic mean $M(a, b, c)$ such that $(M, m)_2$. The Theorem above takes care of the symmetric part, and letting $M(a, b, c) = n(m(a, b), c)$ for all $(a, b, c) \in R_+^3$, with n analytic, forces M to be analytic. However, then M is not symmetric in general. It is satisfying both conditions that does not seem easy to do. Now in certain special cases it is clear how to choose n so that m is both symmetric and analytic. However, these choices do not seem to generalize. For example, if $m(a, b) = \frac{a+b}{2}$, then one can choose $n(a, b) = \frac{2}{3}a + \frac{1}{3}b$, which implies that $n(m(a, b), c) = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$, which of course is the arithmetic mean in three variables. Similarly, if $m(a, b) = \sqrt{ab}$, then one can choose $n(a, b) = a^{2/3}b^{1/3}$, which implies that $n(m(a, b), c) = \sqrt[3]{abc}$. The appearance of the $\frac{2}{3}$ and $\frac{1}{3}$ in each case is not an accident. It is easy to show that a necessary condition for the M from (30) above to be differentiable is $n_y(a, a) = \frac{1}{2}n_x(a, a)$. Also, note that in each of these cases, $n(a, b) = (m \otimes_a m)(b, a)$, where $m \otimes_a n$ denotes the Archimedean compound of m with n (see [7]). One might be tempted to try this approach in general, but it does not work. For example, if $m(a, b) = \frac{a^2 + b^2}{a + b}$ and $n(a, b) = (m \otimes_a m)(b, a)$, then $n(m(1, 2), 3) \approx 2.307961$ and $n(m(1, 3), 2) \approx 2.356566$. Hence if $M(a, b, c) = n(m(a, b), c)$ for all*

$(a, b, c) \in R_+^3$, then M is not symmetric. Of course there is a mean $M(a, b, c)$ such that $(M, m)_2$ -namely $M(a, b, c) = \frac{a^2 + b^2 + c^2}{a + b + c}$.

7.4 Invariance of Both Types

It is also interesting to consider means $M(a, b, c)$ and $m(a, b)$ with M **both** type 1 and type 2 invariant with respect to m . By (16), $M_{xxx}(Q) = \frac{1}{27} (32(f''(1))^2 - 32f''(1))$, $Q = (1, 1, 1)$. Substituting into (29) yields

$$M_{xxx}(Q) = -\frac{8}{9} \left(\frac{2}{3} f^{(iv)}(1) + \frac{8}{3} (f''(1))^3 - \frac{4}{9} (f''(1))^2 - \frac{20}{9} f''(1) \right) \quad (31)$$

$$\text{Now let } m(a, b) = L(a, b) = \frac{b - a}{\ln b - \ln a}, \quad f(x) = \frac{x - 1}{\ln x}$$

By (31), if M is **both** type 1 and type 2 invariant with respect to $m(a, b)$, then

$$\begin{aligned} M_{xxx}(Q) &= -\frac{8}{9} \lim_{x \rightarrow 1} \left(\frac{2}{3} f^{(iv)}(x) + \frac{8}{3} (f''(x))^3 - \frac{4}{9} (f''(x))^2 - \frac{20}{9} f''(x) \right) = \\ &\frac{248}{3645} \\ &\approx 6.80381495 \times 10^{-2} \end{aligned}$$

By (19), if $M(a, b, c) = L_3(a, b, c)$, the invariant logarithmic mean, which is type 1 invariant with respect to $m(x, y)$, then

$$\begin{aligned} M_{xxx}(Q) &= \frac{8}{15} \lim_{x \rightarrow 1} \left(-\frac{2}{3} f^{(iv)}(x) - \frac{8}{9} (f''(x))^3 + \frac{56}{27} (f''(x))^2 + \frac{58}{27} f''(x) \right) = \\ &\frac{136}{2025} \\ &\approx 6.716049 \times 10^{-2} \end{aligned}$$

Since the two values are not equal, L_3 is **not** type 2 invariant with respect to $L(x, y) = \frac{x - y}{\ln x - \ln y}$. In fact, we believe the following is true.

Conjecture 29 Let $m(a, b) = \frac{a + b}{2}$ and $M(a, b, c) = \frac{a + b + c}{3}$. Then the only pairs of means $\{\overline{m}(a, b), \overline{M}(a, b, c)\}$ which satisfy **both** $(\overline{M}, \overline{m})_1$ and $(\overline{M}, \overline{m})_2$ are the means $\overline{m}(a, b) = h^{-1}m(h(a), h(b))$, $\overline{M}(a, b, c) = h^{-1}M(h(a), h(b), h(c))$, where $h(u)$ is a function monotonic on $(0, \infty)$.

8 Open Questions and Future Reserach

(1) One can, of course, define invariance for **nonsymmetric** means. For example, if $m(a, b) = \frac{2}{3}a + \frac{1}{3}b$ and $M(a, b, c) = \frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c$, then

$$M(a, b, m(a, b)) = \frac{2}{3}a + \frac{1}{3}b = m(a, b) \text{ and}$$

$M(m(a, b), m(a, c), m(b, c)) = \frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c = M(a, b, c)$. Hence $(M, m)_1$ and $(M, m)_2$. Note, however, that

$$M(m(a, c), m(a, b), m(b, c)) = \frac{4}{7}a + \frac{5}{21}c + \frac{4}{21}b \neq m(a, b).$$

(2) One might discuss invariance of types 1 and 2 for classes of functions other than means. For example, polynomials or rational functions. One can explore questions such as: What are the invariant polynomials or rational functions in three variables ?

For Type 1 Invariance:

(3) Given any symmetric mean $m(a, b)$, is there always a mean $M(a, b, c)$ such that $(M, m)_1$? We proved this with the additional assumptions that m is strict and isotone.

(4) As discussed earlier, show that $U_0(a, b, c) \leq L_3(a, b, c) \leq U_1(a, b, c)$ for all $(a, b, c) \in R_+^3$, where L_3 is the invariant logarithmic mean and U_0 and U_1 are Stolarsky's generalizations of the logarithmic mean $L(a, b)$.

For Type 2 Invariance:

(5) As discussed earlier, given a symmetric analytic mean $m(a, b)$, is there always a symmetric analytic mean $M(a, b, c)$ such that $(M, m)_2$? In particular, is there an analytic mean $M(a, b, c)$ which is Type 2 invariant with respect to the logarithmic mean $L(a, b)$.

(6) Given a symmetric isotone mean $M(a, b, c)$, and means $m_1(a, b)$ and $m_2(a, b)$ with $(M, m_1)_2$ and $(M, m_2)_2$, must $m_1 = m_2$? We proved this with additional assumptions on M and/or m .

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